Definitions and key facts for section 1.2

The **leading entry** of a (nonzero) row is the leftmost nonzero entry.

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form**

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form).

Fact: Uniqueness of the reduced echelon form
Each matrix is row equivalent to one and only one reduced echelon matrix.

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

A **pivot column** is a column of A that contains a pivot position.

A **pivot** is a nonzero number in a pivot position.

The **row reduction algorithm** produces an echelon matrix in the first four steps and produces a reduced echelon matrix.

Step 1: Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

Step 2: Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Step 3: Use row replacement operations to create zeros in all positions below the pivot.

Step 4: Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

Step 5: Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

The first four steps are the **forward phase** of the row reduction algorithm.

The fifth step is the **backward phase** of the row reduction algorithm.

The variables corresponding to the pivot columns of the reduced echelon form of the augmented matrix of a linear system are **basic variables**. All other variables are **free variable**.

A **parametric description** of the solution set is one in which basic variables are given in terms of the free variables.

Existence and uniqueness theorem
A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. That is, if and only if an echelon form of the augmented matrix has no row of the form

$$\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix} \quad \text{with } b \text{ nonzero.}$$

If a linear system is consistent, then the solution set contains either

1. a unique solution, when there are no free variables, or
2. infinitely many solutions, when there is at least one free variable.